Laplace Transform and Transfer functions

Zero-input & Zero-state Responses

- Let's think about where the terms come from: $$\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y(t)=\frac{dx}{dt}+x(t)$$

$$(s^2+5s+6)Y(s)-\underbrace{(2s+11)}_{\text{Initial condition term}}=\underbrace{\frac{s+1}{s+4}}_{\text{Input term}}$$

$$Y(s)=\underbrace{\frac{2s+11}{s^2+5s+6}}_{\text{zero-input component}}+\underbrace{\frac{s+1}{(s+4)(s^2+5s+6)}}_{\text{zero-state component}}$$

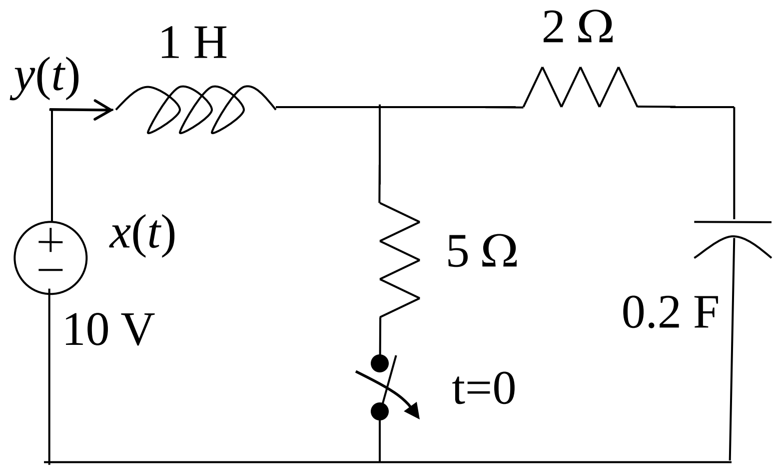
$$=\left[\frac{7}{s+2}-\frac{5}{s+3}\right]+\left[\frac{-1/2}{s+2}+\frac{2}{s+3}-\frac{3/2}{s+4}\right]$$

$$y(t)=\underbrace{\left(7e^{-2t}-5e^{-3t}\right)u(t)}_{\text{zero-input response}}+\underbrace{\left(-\tfrac{1}{2}e^{-2t}+2e^{-3t}-\tfrac{3}{2}e^{-4t}\right)u(t)}_{\text{zero-state response}}$$

Where is H(s)?

Example

In the circuit, the switch is in the closed position for a long time before t=0, when it is opened instantaneously. Find the inductor current y(t) for t $\geq$ 0.



$$L\frac{dy}{dt} + Ry(t) + \frac{1}{C}\int_{-\infty}^{t} y(\tau)d\tau = 10u(t)$$

$$sY(s) - y(0^-) + 2Y(s) + \frac{5Y(s)}{s} + \frac{5\int_{-\infty}^{0^-} y(\tau)d\tau}{s} = \frac{10}{s}$$

$$y(0^-) = \frac{10}{5} = 2A \qquad \int_{-\infty}^{0^-} y(\tau)d\tau = q_c(0^-) = CV = 2$$

$$sY(s) - y(0^-) + 2Y(s) + \frac{5Y(s)}{s} + \frac{10}{s} = \frac{10}{s}$$

$$y(t) = \sqrt{5}e^{-t}\cos(2t + 26.6^o)u(t)$$

Laplace Tranform and Transfer Function

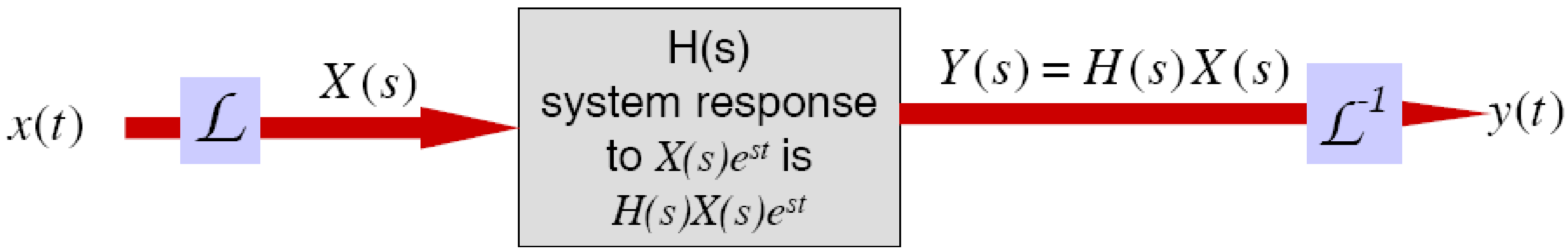
- Let's express input x(t) as a linear combination of exponentials e^{st}:

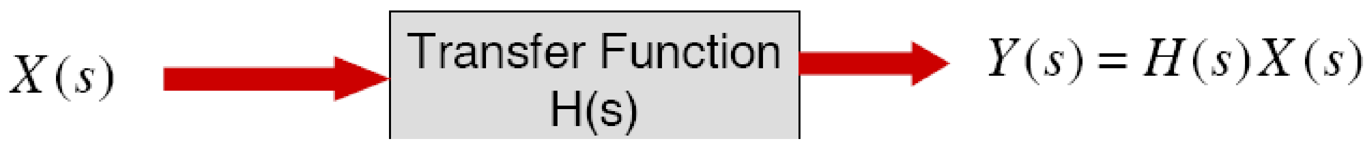
$$x(t) = \sum_{i=1}^{K} X(s_i) e^{s_i t}$$

- H(s) can be regarded as the system's response to each of the exponential components, in such a way that the output y(t) is:

$$y(t) = \sum_{i=1}^{K} X(s_i) H(s_i) e^{s_i t}$$

- Therefore, we get $Y(s) = H(s)\, X(s)$





Example

Find the response y(t) of an LTIC system described by the equation

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=\frac{dx(t)}{dt}+x(t)$$

if the input $x(t) = 3e^{-5t}u(t)$ and all the initial conditions are zero; that is the system is in the zero state (relaxed).

Answer :

$$y(t)=(-2e^{-5t}-e^{-2t}+3e^{-3t})u(t)$$

Transfer Function Examples

$x(t)$ → Delay by T sec → $y(t) = x(t-T)$

$\Updownarrow$ $\Updownarrow$ $\Updownarrow$

$X(s)$ $\qquad H(s) = Y(s) / X(s) = e^{-sT}$ $\qquad Y(s) = X(s)e^{-sT}$

Shifting Property

$x(t)$ → Differentiator d/dt → $y(t) = dx/dt$

$\Updownarrow$ $\Updownarrow$ $\Updownarrow$

$X(s)$ $\qquad H(s) = s$ $\qquad Y(s) = sX(s)$

Differentiation Property

$x(t)$ → Integrator → $y(t) = \int_0^t x(\tau)d\tau$

$\Updownarrow$ $\Updownarrow$ $\Updownarrow$

$X(s)$ $\qquad H(s) = \frac{1}{s}$ $\qquad Y(s) = \frac{1}{s}X(s)$

Integration Property

Internal Stability

- Internal Stability (Asymptotic)
  - If and only if all the poles are in the LHP
  - Unstable if, and only if, one or both of the following conditions exist:
    - At least one pole is in the RHP
    - There are repeated poles on the imaginary axis
  - Marginally stable if, and only if, there are no poles in the RHP, and there are some unrepeated poles on the imaginary axis.

External Stability BIBO

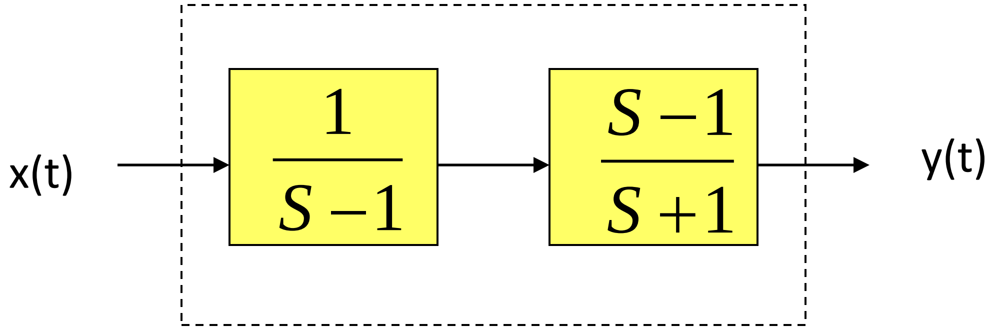
The transfer function H(s) can only indicate the external stability of the system BIBO.

$$H(s) = \frac{b_0 s^M + b_1 s^{M-1} + \ldots + b_M}{s^N + a_1 s^{N-1} + \ldots + a_N}$$

BIBO stable if $M \leq N$ and all poles are in the LHP

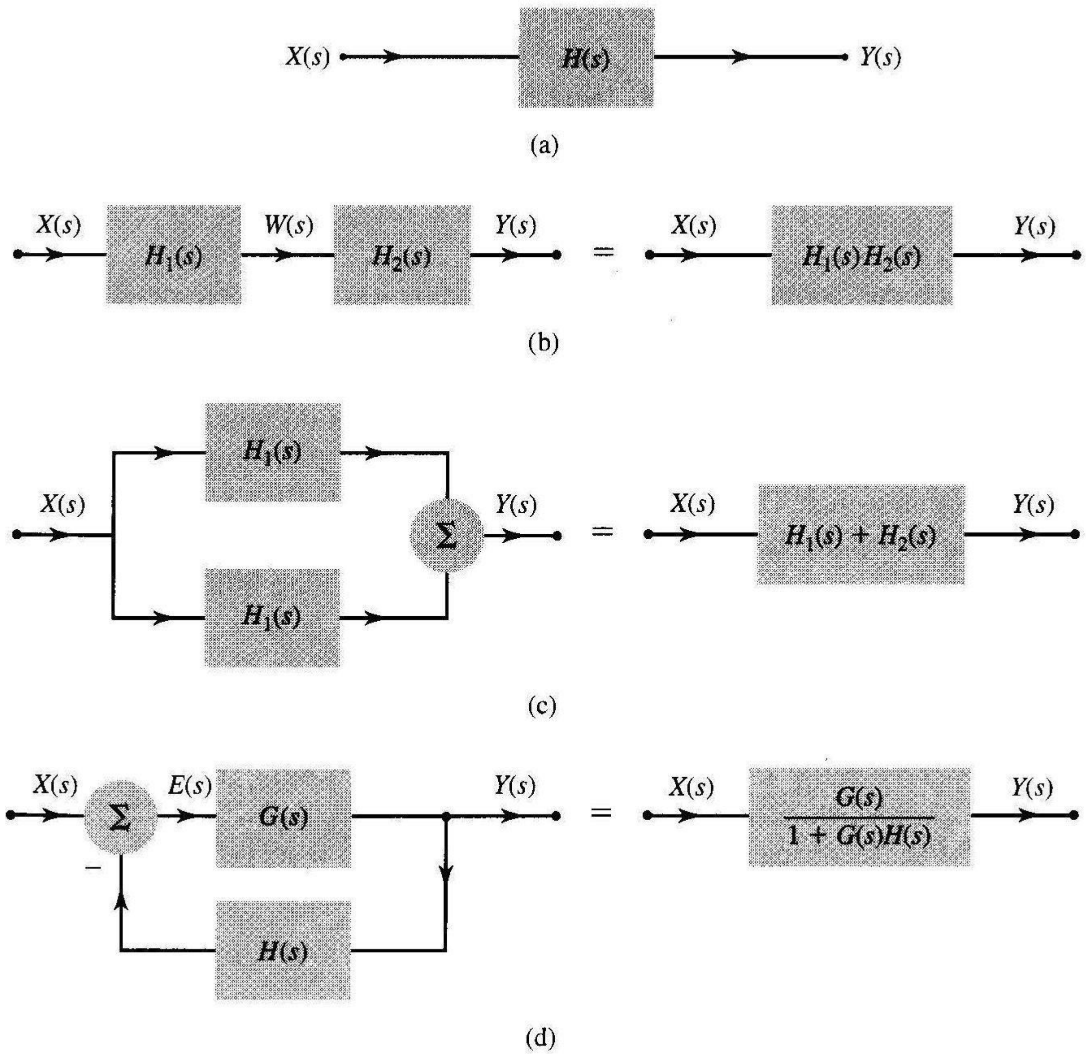
Example

Is the system below BIBO and asymptotically (internally) stable?



Block Diagrams

$$\frac{Y(s)}{X(s)} = \frac{W(s)}{X(s)}\frac{Y(s)}{W(s)} = H_1(s)H_2(s)$$



System Realization

$$H(s)=\frac{b_0 s^{M}+b_1 s^{M-1}+\ldots+b_M}{s^{N}+a_1 s^{N-1}+\ldots+a_N}$$

- Realization is a synthesis problem, so there is no unique way of realizing a system.
- A common realization of H(s) is using
  - Integrator
  - Scalar multiplier
  - Adders

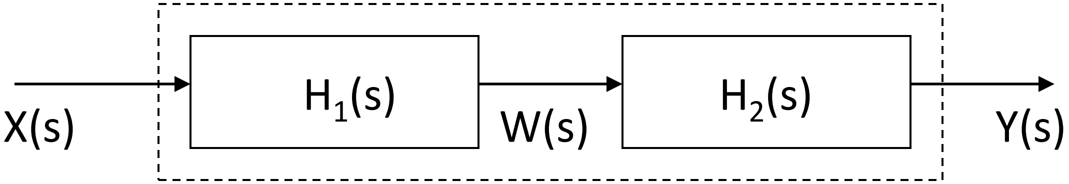
Direct Form I Realization

$$H(s)=\frac{b_0 s^3+b_1 s^2+b_2 s+b_3}{s^3+a_1 s^2+a_2 s+a_3}$$

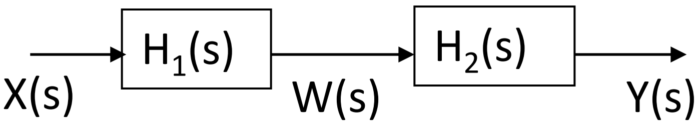
Divide every term by s with the highest order s^3

$$H(s)=\frac{b_0+\frac{b_1}{s}+\frac{b_2}{s^2}+\frac{b_3}{s^3}}{1+\frac{a_1}{s}+\frac{a_2}{s^2}+\frac{a_3}{s^3}}$$

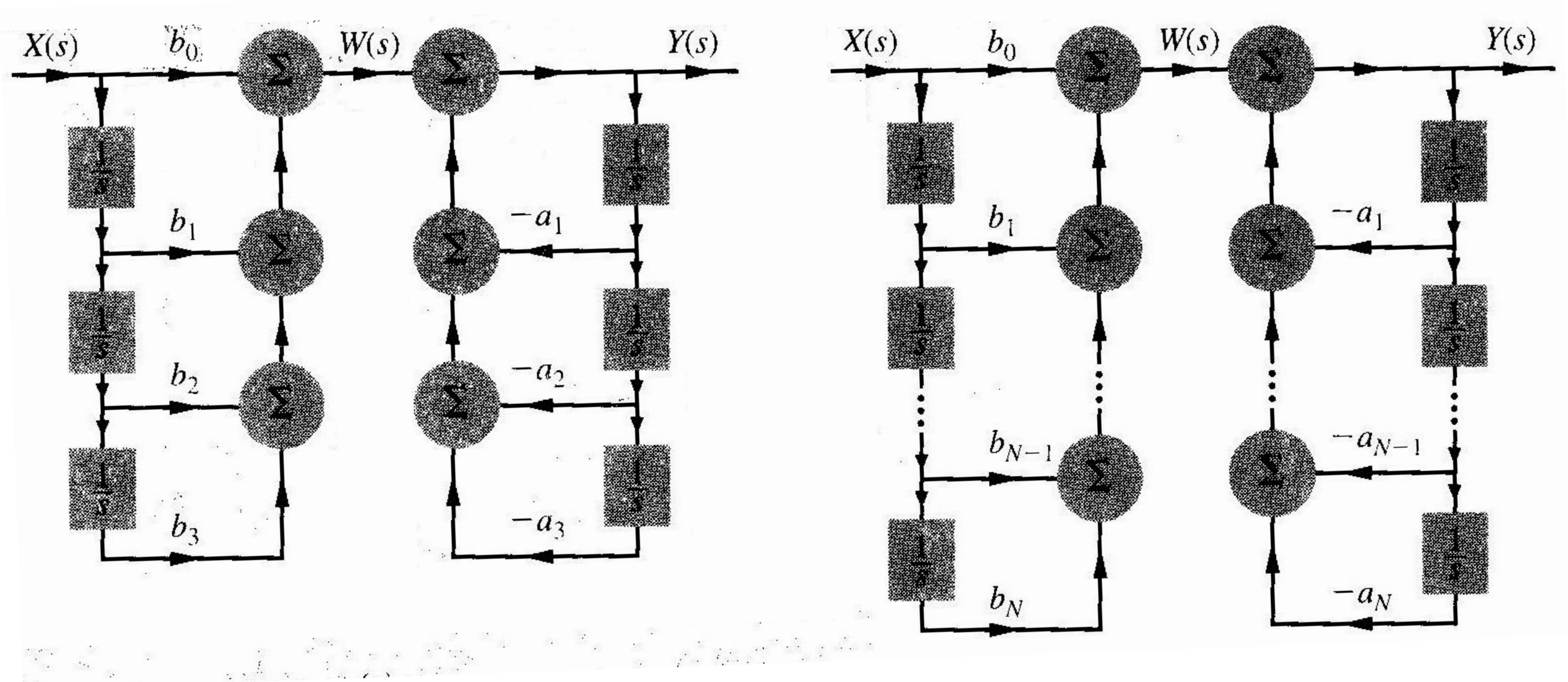
$$H(s)=\left(b_0+\frac{b_1}{s}+\frac{b_2}{s^2}+\frac{b_3}{s^3}\right)\left(\frac{1}{1+\frac{a_1}{s}+\frac{a_2}{s^2}+\frac{a_3}{s^3}}\right)$$



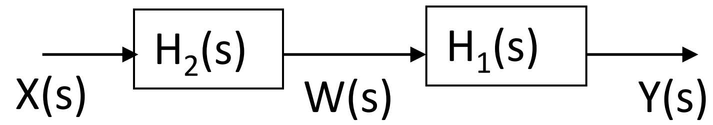
Direct Form I Realization



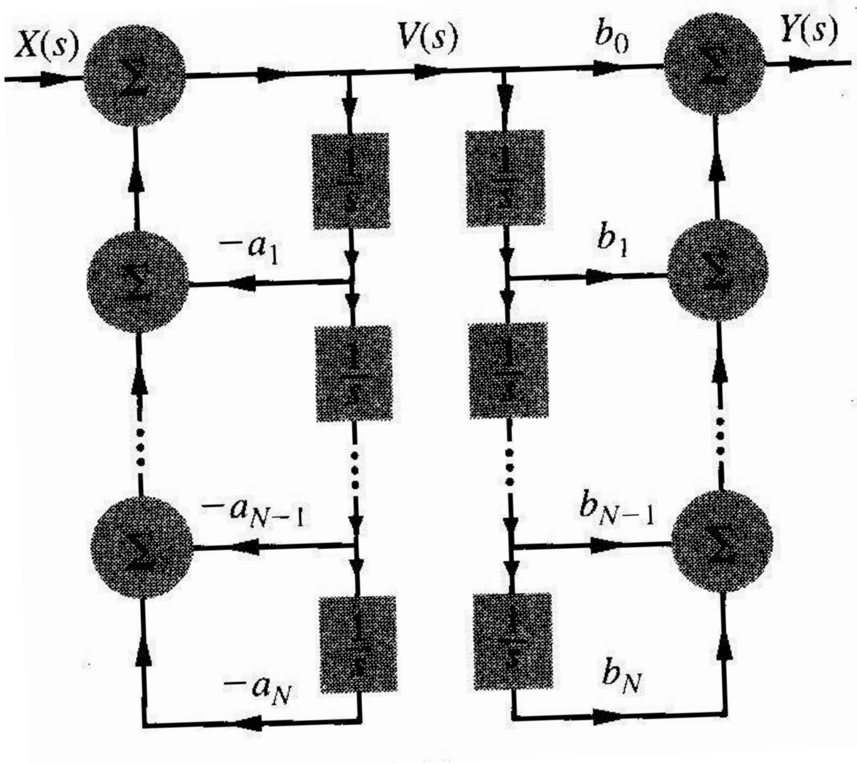
$$H(s) = \left(b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3} \right) \left(\frac{1}{1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}} \right)$$

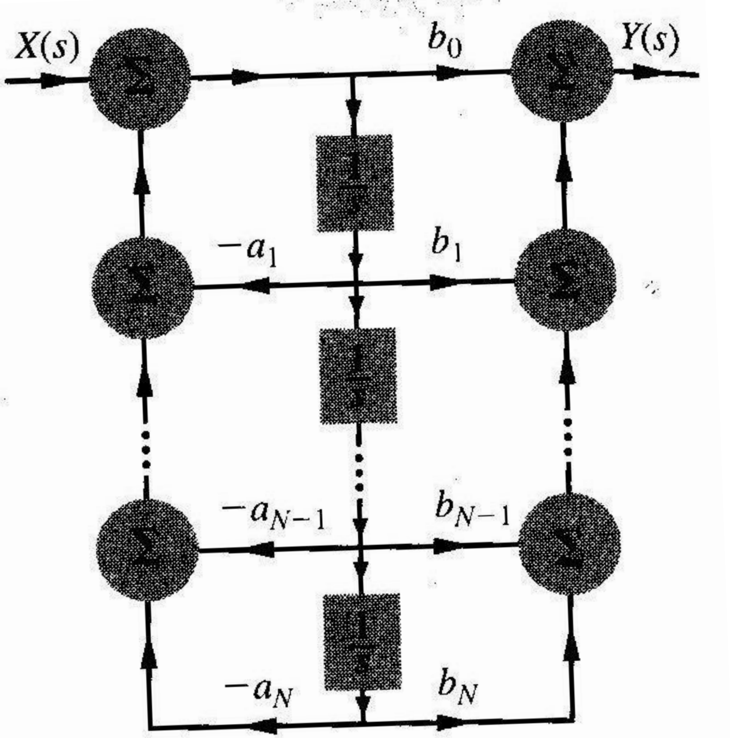


Direct Form II Realization



$$H(s)=\left(\frac{1}{1+\frac{a_1}{s}+\frac{a_2}{s^2}+\frac{a_3}{s^3}}\right)\left(b_0+\frac{b_1}{s}+\frac{b_2}{s^2}+\frac{b_3}{s^3}\right)$$





Example

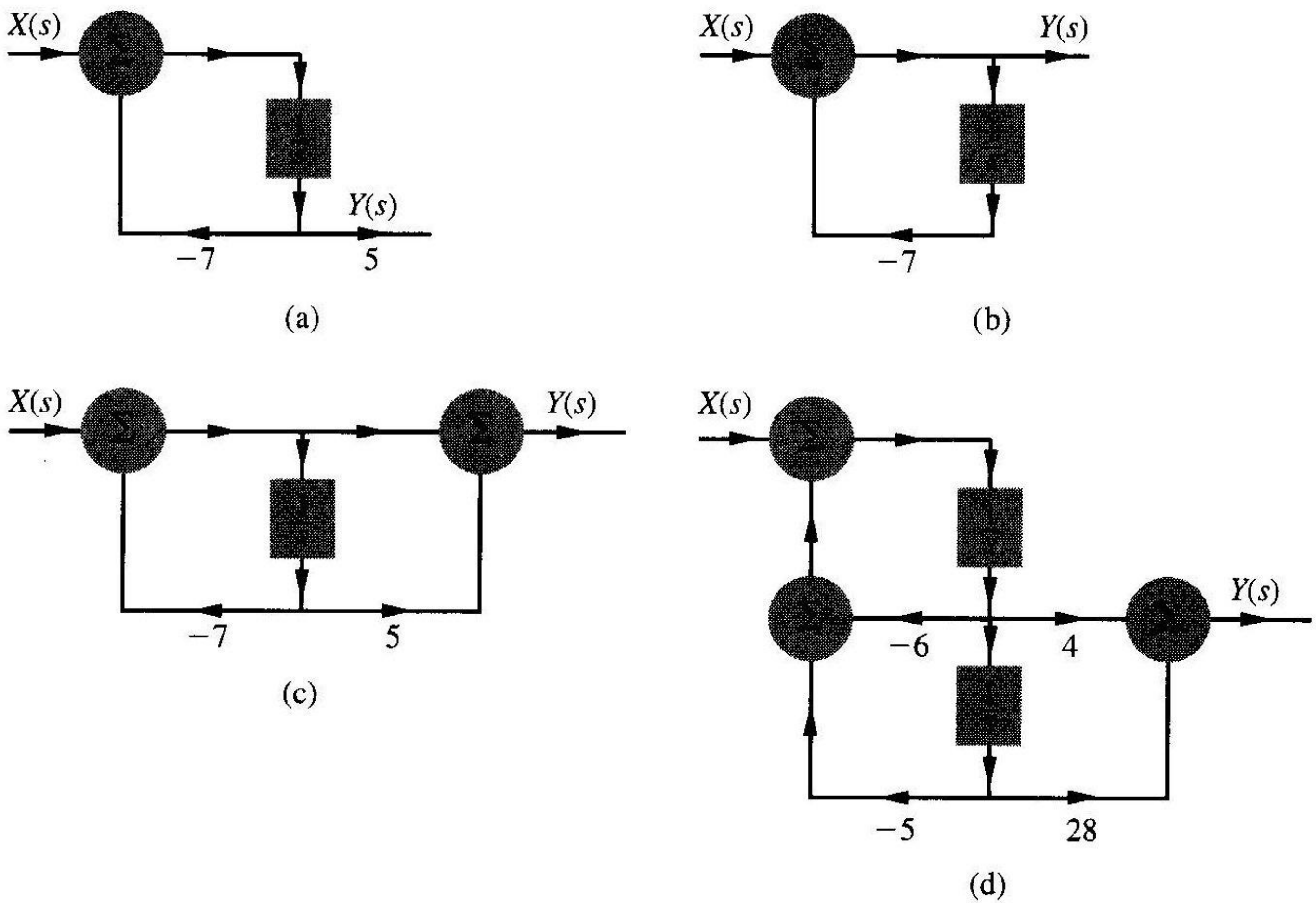
Find the canonic direct form realization of the following transfer functions:

a) $\dfrac{5}{s+7}$

b) $\dfrac{s}{s+7}$

c) $\dfrac{s+5}{s+7}$

d) $\dfrac{4s+28}{s^2+6s+5}$



Cascade and Parallel Realizations

$$H(s) = \frac{4s+28}{s^2+6s+5}$$

Cascade Realization

$$H(s) = \frac{4s+28}{(s+1)(s+5)} = \left(\frac{4s+28}{s+1}\right)\left(\frac{1}{s+5}\right)$$

Parallel Realization

$$H(s) = \frac{4s+28}{(s+1)(s+5)} = \frac{6}{s+1} - \frac{2}{s+5}$$

The complex poles in H(s) should be realized as a second-order system.